Double Integrals over General Regions

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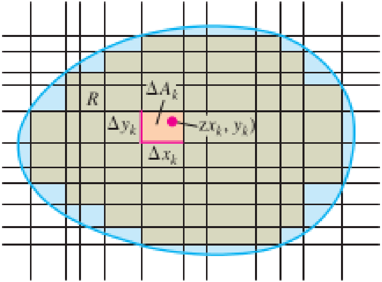
Overview

The nature of the boundary of R introduces issues not found in integrals over an interval. When R has a curved boundary, the n rectangles of a partition lie inside R but do not cover all of R. In order for partition to approximate R well, the parts of R covered by small rectangles lying partly outside R must become negligible as the norm of the partition approaches zero. This property of being nearly filled in by a partition of small norm is satisfied by all the regions that we will encounter.

There is no problem with boundaries made from polygons, circles, ellipse, and from continuous graphs over an interval, joined end to end. A curve with a "fractal" type of shape would be problematic, but such curves arise rarely in most applications. A careful discussions of which type of regions R can be used for computing double integrals is left to a more advanced text.

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region R, we begin by covering R with a grid of small rectangular cells whose union contains all points of R.



This time, however, we cannot exactly fill R with a finite number of rectangles in the grid lie partly outside R. A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside.

Double Integrals over Bounded Nonrectangular Regions

For commonly arising regions, more and more of R is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero. Once we have a partition of R, we number the rectangles in some order from 1 to n and let ΔA_k be the area of the kth rectangle. We then choose a point (x_k, y_k) in the kth rectangle and form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(c_k, y_k)\Delta A_k.$$

Double Integrals over Bounded Nonrectangular Regions

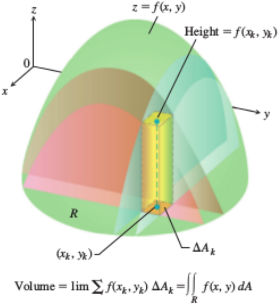
As the norm of the partition forming S_n goes to zero, $\|P\| \to 0$, the width and height of each enclosed rectangle goes to zero and their number goes to infinity.

If $f(x, y)$ is a continuous function, then these Riemann sums converge to a limiting value, not dependent on any of the choices we made. This limit is callld the **double integral** of $f(x, y)$ over R :

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k = \iint\limits_{R} f(x, y) \, dA.$$

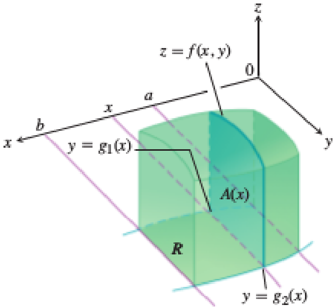
If $f(x, y)$ is positive and continuous over R we define the volume of the solid generated between R and the surface $z = f(x, y)$ to be

$$\iint_R f(x, y)\, dA.$$



Double Integrals over Bounded Nonrectangular Regions

If R is a region like the one shown in the xy-plane in the following figure, bounded "above" and "below" by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a$, $x = b$, we may again calculate the volume by the method of slicing.



Double Integrals over Bounded Nonrectangular Regions

We first calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\ dy$$

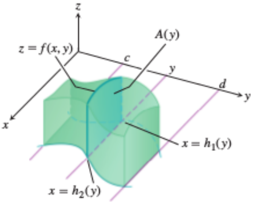
and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$V = \int_a^b A(x)\ dx = \int_a^b \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\ dy\ dx.$$

Double Integrals over Bounded Nonrectangular Regions

Similarly, if R is a region like the one shown in the following figure, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines $y = c,\ y = d$, the the volume calculated by slicing is given by the iterated integral.

$$\text{Volume } = \int_c^d \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy.$$



Fubini's Theorem (Stronger Form)

That the iterated integrals both give the volume that we defined to be the double integral of f over R is a consequence of the following stronger form of Fubini's Theorem.

Theorem 1.

Let $f(x, y)$ be continuous on a region R.

- *If R is defined by $a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then*

$$\iint\limits_R f(x, y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy\ dx.$$

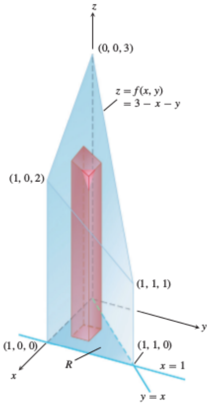
- *If R is defined by $c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then*

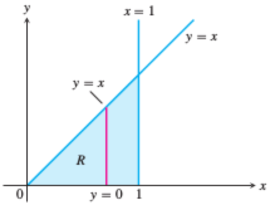
$$\iint\limits_R f(x, y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\ dx\ dy.$$

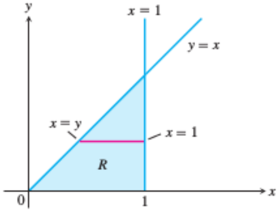
Example 1.

Find the volume of the prism whose base is the triangle in the *xy*-plane bounded by the *x*-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$







Solution

$$V = \int_0^1 \int_0^x (3 - x - y)\ dy\ dx = \int_0^1 \int_y^1 (3 - x - y)\ dxdy.$$

Fubini's Theorem (Stronger Form)

Although Fubini's Theorem assures us that a double integral may be calculated as an iterated integral in either order of integration, the value of one integral may be easier to find than the value of the other.

Example 2.

Calculate

$$\iint_R \frac{\sin x}{x}\ dA$$

where A is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

If we integrate first with respect to y and then with respect to x, we get $-\cos(1) + 1$ as the answer.

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\ dy \right)\ dx = -\cos 1 + 1 \approx 0.46.$$

Fubini's Theorem (Stronger Form)

But if we reverse the order of integration and attempt to calulate, we run into a problem because

$$\int \frac{\sin x}{x}\ dx$$

cannot be expressed in terms of elementary functions (there is no simple antiderivative).

There is no general rule for predicting which order of integration will be good. If the order we first choose doesn't work, we try the other. Sometimes neither order will work, and then we need to use numerical approximations.

Finding Limits of Integration

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

Using Vertical Cross-sections

When faced with evaluation

$$\iint_R f(x, y)\, dA,$$

integrating first with respect to y and then with respect to x, do the following.

Finding Limits of Integration

1. **Sketch :** Sketch the region of integration and label the bounding curves.
2. y **limits :** Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).
3. x **limits :** Choose x-limits that include all the vertical lines through R.

Finding Limits of Integration

Using Horizontal Cross-sections

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in steps 2 and 3.

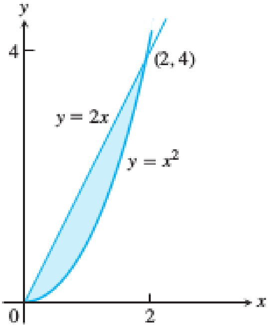
Double integrals of continuous functions over nonrectangular regions have the same algebraic properties as integrals over rectangular regions. These properties are useful in computations and applications.

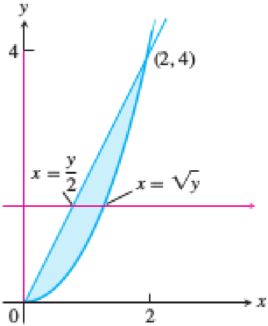
Example 3.

Write an equivalent integral for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\ dy\ dx$$

and write an equivalent integral with the order of integration reversed.





Solution

An equivalent integral with the order of integration reversed is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\ dx\ dy = 8$$

Properties of Double Integrals

If $f(x,y)$ and $g(x,y)$ are continuous, then

- **Constant Multiple :**
$$\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA, \quad \text{any number } c.$$
- **Sum and Difference :**
$$\iint\limits_R \Big\{f(x,y) \pm g(x,y)\Big\}\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA.$$

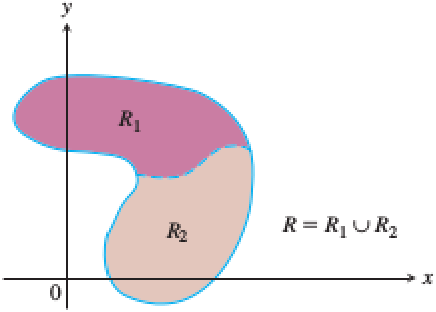
Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous, then

- **Domination :**
  - $\displaystyle\iint\limits_R f(x, y)\ dA \geq 0$ if $f(x, y) \geq 0$ on A.
  - $\displaystyle\iint\limits_R f(x, y)\ dA \geq \iint\limits_R g(x, y)\ dA$ if $f(x, y) \geq g(x, y)$ on R.

Properties of Double Integrals

- **Additivity :** $$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$$
  if R is the union of two nonoverlapping regions R_1 and R_2.



Properties of Double Integrals

The idea behind these properties is that integrals behave like sums. If the function $f(x, y)$ is replaced by its constant multiple $cf(x, y)$, then a Riemann sum for f

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

is replaced by a Riemann sum for cf

$$\sum_{k=1}^{n} cf(x_k, y_k)\Delta A_k = c\sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = cS_n.$$

Properties of Double Integrals

Taking limits as $n \to \infty$ shows that

$$c \lim_{n\to\infty} S_n = c \iint\limits_R f \, dA$$

and

$$\lim_{n\to\infty} cS_n = \iint\limits_R cf \, dA$$

are equal. It follows that the constant multiple property carries over from sums to double integrals.

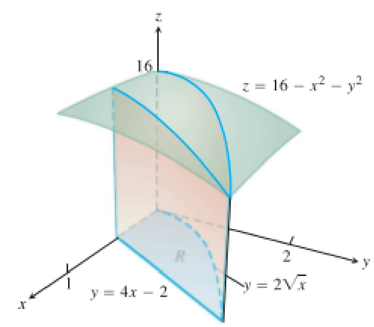
The other properties are also easy to verify for Riemann sums, and carry over to double integrals for the same reason.

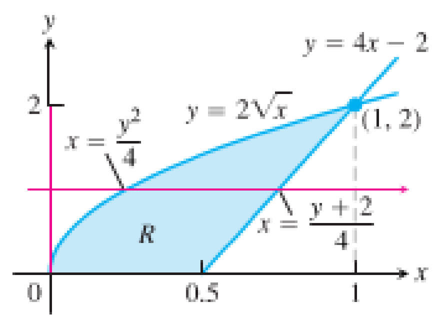
Example 4.

Find the volume of the wedgelike solid that lies beneath the surface

$$z = 16 - x^2 - y^2$$

and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.





Solution

The volume is

$$V = \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\ dx\ dy = \frac{20803}{1680}$$

Exercises

1. State Fubini's Theorem for a continuous function on a region.
2. Write two properties of double integrals.
3. Evaluate
$$\int_{-2}^{2}\int_{\pi/6}^{5\pi/6}\left\{x^3 e^{\cos^2 y} + \sec\left(\frac{x}{2}\right)\cos y\right\}\,dy\ dx.$$
4. Sketch the region of integration, determine the order of integration, and evaluate the integral
   (a) $\int\int_R (y - 2x^2)\ dA$ where R is the region inside the square
   $$|x| + |y| = 1.$$
   (b) $\int_0^3 \int_{\sqrt{x/3}}^{1} e^{y^3} dy\ dx.$

Solutions

4. (a) $$\int\int_R (y-2x^2)\ dA = \int_{-1}^{0}\int_{-x-1}^{x+1}(y-2x^2)\ dy\ dx = \int_0^1\int_{x-1}^{1-x}(y-2x^2)\ dy\ dx = -\frac{2}{3}$$

   (b) $$\int_0^3\int_{\sqrt{x/3}}^{1} e^{y^3}\ dy\ dx = \int_0^1\int_0^{3y^2} e^{y^3}\ dx\ dy = e-1$$

Exercises

5. Using double integral find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.
6. Sketch the region of integration, write an equivalent double integral with order of integration reversed and evaluate the integral:

$$\int_0^1 \int_y^1 x^2 e^{xy} \, dx \, dy.$$

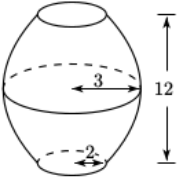
7. Evaluate the improper integral

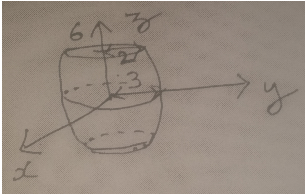
$$\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y} \, dy \, dx.$$

Solutions

5. $V = \int_0^2 \int_0^{2-x} (12 - 3y^2)\ dy\ dx = 20$
6. $\int_0^1 \int_y^1 x^2 e^{xy}\ dx\ dy = \int_0^1 \int_0^x x^2 e^{xy}\ dy\ dx = \dfrac{e-2}{2}$
7. $\int_1^\infty \int_{e^{-x}}^1 \dfrac{1}{x^3 y}\ dy\ dx = \int_1^\infty \left[\dfrac{\ln y}{x^3}\right]_{e^{-x}}^1\ dx = -\lim_{b\to\infty} \left[\dfrac{1}{x}\right]_1^b = 1$

8. A symmetrical urn holds 24 buckets of water when it is full. The interior has a circular cross-section whose radius reduces from 3 m at the centre to 2m at the base and top. The height (between base and top) of the urn is 12 m. The bounding surface of the urn is generated by revolving a parabola. When 6 buckets of water is stored, what would be the level of water (in the urn) from the bottom ?



8. 

The equation of the parabola is $y = -\left(\frac{z}{6}\right)^2 + 3$. It is given that

$$\int_{z=-6}^{z_0} \pi\left[3 - \left(\frac{z^2}{36}\right)^2\right]\,dz = \frac{1}{4}\int_{z=-6}^{6} \pi\left[3 - \left(\frac{z^2}{36}\right)^2\right]\,dz.$$

We get the relation

$$9z_0 - \frac{z_0^3}{18} + \frac{z_0^5}{5 \times 36^2} = -21.60.$$

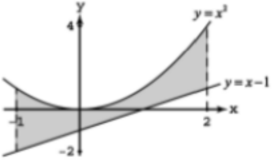
Hence the level of the water is $z_0 + 6$ m, where z_0 satisfies the above relation. Note that z_0 is negative and it is 3.51 m.

Exercise 2.

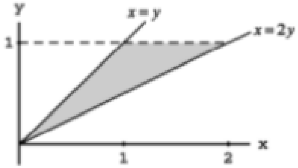
In the following exercises, sketch the described regions of integration.

1. $-1 \leq x \leq 2, \quad x - 1 \leq y \leq x^2$
2. $0 \leq y \leq 1, \quad y \leq x \leq 2y$
3. $1 \leq x \leq e^2, \quad 0 \leq y \leq \ln x$
4. $0 \leq y \leq 1, \quad 0 \leq x \leq sin^{-1} y$
5. $0 \leq y \leq 8, \quad \frac{1}{4}y \leq x \leq y^{\frac{1}{3}}$

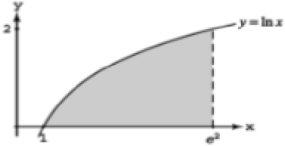
Solution for (1.) in Exercise 2



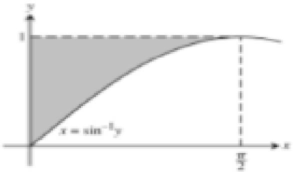
Solution for (2.) in Exercise 2



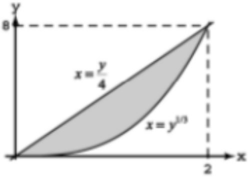
Solution for (3.) in Exercise 2



Solution for (4.) in Exercise 2



Solution for (5.) in Exercise 2



Exercise 3.

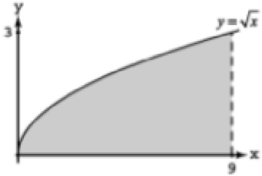
In the following exercises, write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

1. *Bounded by* $y = \sqrt{x}, y = 0,$ *and* $x = 9$
2. *Bounded by* $y = e^x, y = 1,$ *and* $x = \ln 3$
3. *Bounded by* $y = 3 - 2x, y = x,$ *and* $x = 0$
4. *Bounded by* $y = x^2$*and* $y = x + 2$

Solution for (1.) in Exercise 3

(a) $\displaystyle\int_0^9 \int_0^{\sqrt{3}} dy\ dx$

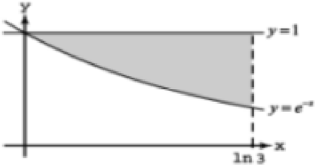
(b) $\displaystyle\int_0^3 \int_{y^2}^{9} dx\ dy$



Solution for (2.) in Exercise 3

(a) $\displaystyle\int_0^{\ln 3}\int_{e^{-x}}^{1} dy\ dx$

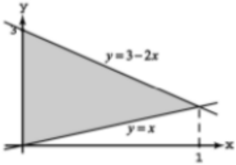
(b) $\displaystyle\int_{1/3}^{1}\int_{-\ln y}^{\ln 3} dx\ dy$



Solution for (3.) in Exercise 3

(a) $\displaystyle\int_0^1 \int_x^{3-2x} dy\ dx$

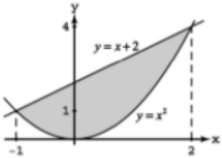
(b) $\displaystyle\int_0^1 \int_0^y dx\ dy + \int_1^3 \int_0^{(3-y)/2} dx\ dy$



Solution for (4.) in Exercise 3

(a) $$\int_{-1}^{2}\int_{x^2}^{x+2} dy\ dx$$

(b) $$\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\ dy + \int_{1}^{3}\int_{y-2}^{\sqrt{y}} dx\ dy$$



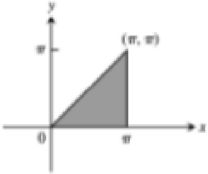
Exercise 4.

In the following exercises, sketch the region of integration and evaluate the integral.

1. $\displaystyle\int_0^{\pi}\int_0^{x} x \ \sin y \ dy \ dx$
2. $\displaystyle\int_1^{2}\int_y^{y^2} dx \ dy$
3. $\displaystyle\int_0^{1}\int_0^{y^2} 3y^3 e^{xy} dx \ dy$

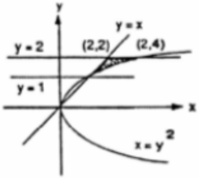
Solution for (1.) in Exercise 4

$$\int_0^{\pi}\int_0^{x}(x\sin y)dy\ \ dx = \int_0^{x}\Big[-x\cos y\Big]_0^{x}dx$$
$$= \int_0^{\pi}(x - x\cos x)dx$$
$$= \left[\frac{x^2}{2} - (\cos x + x\sin x)\right]_0^{\pi} = \frac{\pi^2}{2} + 2$$



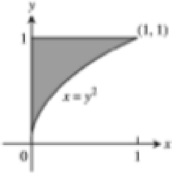
Solution for (2.) in Exercise 4

$$\begin{aligned}\int_1^2 \int_y^{y^2} dx\ dy &= \int_1^2 (y^2 - y)dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2 \\ &= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}\end{aligned}$$



Solution for (3.) in Exercise 4

$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\;dy = \int_0^1 \left[3y^2 e^{xy}\right]_0^{y^2} dy$$
$$= \int_0^1 (3y^2 e^{y^3} - 3y^2)dy = \left[e^{y^3} - y^3\right]_0^1 = e - 2$$



Exercise 5.

In the following exercises, integrate f over the given region.

1. Quadrilateral : $f(x, y) = x/y$ *over the region in the first quadrant bounded by the lines* $y = x, y = 2x, x = 1,$ *and* $x = 2$
2. Triangle : $f(x, y) = x^2 + y^2$ *over the triangular region with vertices* $(0, 0), (1, 0),$ *and* $(0, 1)$
3. Triangle : $f(u, v) = v - \sqrt{u}$ *over the triangular region cut from the first quadrant of the* $uv-$ *plane by the line* $u + v = 1$
4. Curved region : $f(s, t) = e^x \ln t$ *over the region in the first quadrant of the* $st-$ *plane that lies above the curve* $s = \ln t$ *from* $t = 1$ *to* $t = 2$.

Solution for Exercise 5

1. $\int_1^2 \int_x^{2x} \frac{x}{y} dy\ dx = \int_1^2 [x \ln\ y]_x^{2x} dx\ = (\ln\ 2) \int_1^2 x\ dx = \frac{3}{2} \ln 2$
2. $\int_0^1 \int_0^{1-x} (x^2 + y^2) dy\ dx = \int_0^1 [x^2 y + \frac{y^0}{3}]_0^{1-x} dx =$
   $\int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx = \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx =$
   $\left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} - 0\right) - \left(0 - 0 - \frac{1}{12}\right) = \frac{1}{6}$
3. $\int_0^1 \int_0^{1-u} (v - \sqrt{u}) dv\ du = \int_0^1 \left[\frac{v^2}{2} - v\sqrt{u}\right]_0^{1-u} du =$
   $\int_0^1 \left[\frac{1-2u+u^2}{2} - \sqrt{u}(1-u)\right] du = \int_0^1 \left(\frac{1}{2} - u + \frac{u^2}{2} - u^{1/2} + u^{3/2}\right) du =$
   $\left[\frac{u}{2} - \frac{u^2}{2} + \frac{u^3}{6} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{2} + \frac{2}{5} = -\frac{1}{16}$
4. $\int_1^2 \int_0^{\ln r} e^5 \ln\ t\ ds\ dt = \int_1^2 [e^5 \ln\ t]_0^{\ln t} dt = \int_1^2 (t\ \ln\ t\ - \ln\ t) dt =$
   $\left[\frac{t^2}{2} \ln\ t - \frac{t^2}{4} - t\ \ln\ t + t\right]_1^2 = (2 \ln 2 - 1 - 2 \ln 2 + 2) - (-\frac{1}{4} + 1) = \frac{1}{4}$

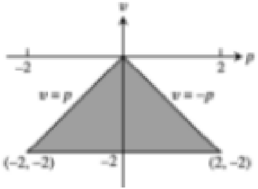
Exercise 6.

Each of the following exercises gives an integral over a region in a Cartesian coordinate plane. Sketch the region and evaluate the integral.

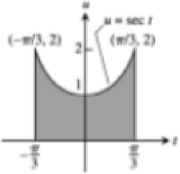
1. $\displaystyle\int_2^0 \int_v^{-v} 2 \; dp \; dv \quad$ *(the pv-plane)*
2. $\displaystyle\int_{\pi/3}^{\pi/3} \int_0^{\sec 1} 3\cos \; t \; du \; dt$ *(the tu− plane)*
3. $\displaystyle\int_0^{3/2} \int_1^{4-2u} \frac{4-2u}{v^2} dv \; du$ *(the uv−plane)*

Solution for (1.) in Exercise 6

$$\begin{aligned}
\int_{-2}^{0}\int_{v}^{-v} 2dp\ dv &= 2\int_{-2}^{0}[p]_{v}^{-v}dv \\
&= 2\int_{-2}^{0} -2v\ dv \\
&= -2[v^2]_{-2}^{0} = 8
\end{aligned}$$

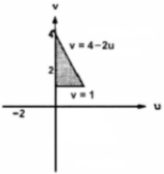


$$\int_{-x/3}^{x/3}\int_{0}^{\sec t} 3\ \cos\ t\ du\ dt = \int_{-x/3}^{x/3}[(3\ \cos\ t)u]_0^{\sec\ t}$$
$$= \int_{-x/3}^{x/3} 3\ dt = 2\pi$$



Solution for (3.) in Exercise 6

$$
\begin{aligned}
\int_0^{3/2}\int_1^{4-2u} \frac{4-2u}{v^2}\,dv\ du &= \int_0^{3/2} \left[\frac{2u-4}{v}\right]_1^{4-2u} du \\
&= \int_0^{3/2} (3-2u)du = [3u-u^2]_0^{3/2} = \frac{9}{2}
\end{aligned}
$$



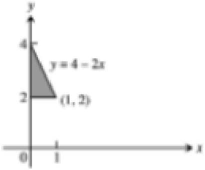
Exercise 7.

In the following exercises, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

1. $\displaystyle\int_0^1 \int_2^{4-2x} dy\ dx$
2. $\displaystyle\int_0^{\ln 2} \int_{e^x}^{2} dx\ dy$
3. $\displaystyle\int_0^{3/2} \int_0^{9-4x^2} 16x\ dy\ dx$
4. $\displaystyle\int_1^{e} \int_0^{\ln x} xy\ dy\ dx$
5. $\displaystyle\int_0^{\sqrt{3}} \int_0^{\tan^{-1} y} \sqrt{xy}\ dx\ dy$

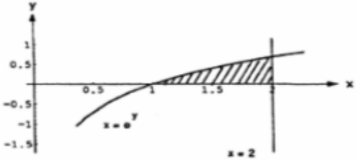
Solution for (1.) in Exercise 7

$$\int_2^4 \int_0^{(4-y)/2} dx\ dy$$



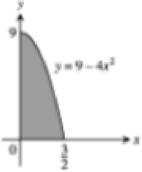
Solution for (2.) in Exercise 7

$$\int_1^2 \int_0^{\ln\ x} dy\ dx$$



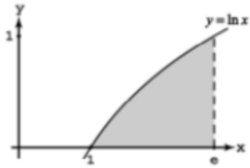
Solution for (3.) in Exercise 7

$$\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x \; dx \; dy$$



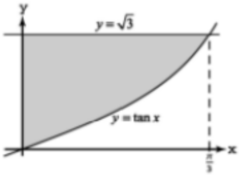
Solution for (4.) in Exercise 7

$$\int_0^1 \int_{e^y}^{e} x\ y\ dx\ dy$$



Solution for (5.) in Exercise 7

$$\int_0^{\pi/3} \int_{\tan\ x}^{\sqrt{3}} \sqrt{xy}\, dy\ dx$$



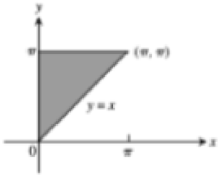
Exercise 8.

In the following exercises, sketch the region of integration, reverse the order of integration, and evaluate the integral.

1. $\displaystyle\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dy \ dx$
2. $\displaystyle\int_0^{2} \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dy \ dx$
3. $\displaystyle\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right) dx \ dy$
4. $\displaystyle\int_0^{8} \int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1} dy \ dx$

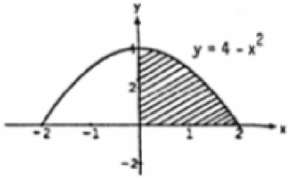
Solution for (1.) in Exercise 8

$$\int_0^{\pi}\int_x^{\pi}\frac{\sin\ y}{y}\,dy\ dx = \int_0^{\pi}\int_0^{y}\frac{\sin\ y}{y}\,dx\ dy$$
$$= \int_0^{\pi}\sin\ y\ dy = 2$$



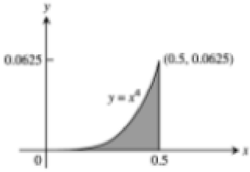
Solution for (2.) in Exercise 8

$$
\begin{aligned}
\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dy\ dx &= \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} dx\ dy \\
&= \int_0^4 [\frac{x^2 e^{2y}}{2(4-y)}]_0^{\sqrt{4-y}} dy \\
&= \int_0^4 \frac{e^{2y}}{2} dy = [\frac{e^{2y}}{4}]_0^4 = \frac{e^8 - 1}{4}
\end{aligned}
$$



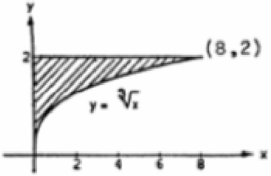
Solution for (3.) in Exercise 8

$$\begin{aligned}
\int_0^{1/16}\int_{y^{4/4}}^{1/2} \cos(16\pi x^5)dx\ dy &= \int_0^{1/2}\int_0^{x^4} \cos(16\pi x^5)dy\ dx \\
&= \int_0^{1/2} x^4 \cos(16\pi x^5)dx \\
&= \Big[\frac{\sin(16\pi x^5)}{80\pi}\Big]_0^{1/2} = \frac{1}{80\pi}
\end{aligned}$$



Solution for (4.) in Exercise 8

$$
\begin{aligned}
\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\,dy\;dx &= \int_0^2 \int_0^{y^3} \frac{1}{y^3+1}\,dx\;dy \\
&= \int_0^2 \frac{y^3}{y^4+1}\,dy \\
&= \frac{1}{4}\Big[\ln(y^4+1)\Big]_0^2 = \frac{\ln\ 17}{4}
\end{aligned}
$$

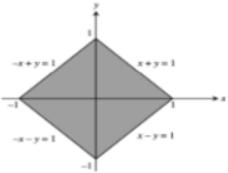


Exercise 9.

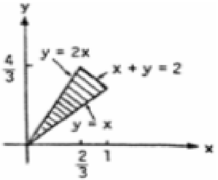
In the following exercises, sketch the region of integration, reverse the order of integration, and evaluate the integral.

1. Square region : $\iint_R (y - 2x^2)\, dA$ *where* R *is the region bounded by the square* $|x| + |y| = 1$
2. Triangular region : $\iint_R xy\, dA$ *where* R *is the region bounded by the lines* $y = x, y = 2x,$ *and* $x + y = 2$

$$
\begin{aligned}
\iint_R (y-2x^2)dA &= \int_{-1}^{0}\int_{-x-1}^{x+1}(y-2x^2)dy\ dx + \int_0^1\int_{x-1}^{1-x}(y-2x^2)dy\ dx \\
&= \int_{-1}^{0}[\frac{1}{2}y^2-2x^2y]_{-x-1}^{x+1}dx + \int_0^1[\frac{1}{2}y^2-2x^2y]_{x-1}^{1-x}dx \\
&= -4\int_{-1}^{0}(x^3+x^2)dx + 4\int_0^1(x^3-x^2)dx \\
&= -4[\frac{x^4}{4}+\frac{x^3}{3}]]_{-1}^{0} + 4[\frac{x^4}{4}-\frac{x^3}{3}]_0^1 = 4[\frac{(-1)^4}{4}+\frac{(-1)^3}{3}] + 4\left(\frac{1}{4}-\frac{1}{3}\right) \\
&= 8\left(\frac{3}{12}-\frac{4}{12}\right) = -\frac{8}{12} = -\frac{2}{3}
\end{aligned}
$$



$$
\begin{aligned}
\iint_R xy\ dA &= \int_0^{2/3}\int_x^{2x} xy\ dy\ dx + \int_{2/3}^{1}\int_x^{2-x} xy\ dy\ dx \\
&= \int_0^{2/3} \left[\frac{1}{2}xy^2\right]_x^{2x} dx + \int_{2/3}^{1} \left[\frac{1}{2}xy^2\right]_x^{2-x} dx \\
&= \int_0^{2/3} (2x^3 - \frac{1}{2}x^3) dx + \int_{2/3}^{1} \left[\frac{1}{2}x(2-x)^2 - \frac{1}{2}x^3\right] dx \\
&= \int_0^{2/3} \frac{3}{2}x^3 dx + \int_{2/3}^{1} (2x - x^2) dx = \left[\frac{3}{8}x^4\right]_0^{2/3} + \left[x^2 - \frac{2}{3}x^3\right]_{2/3}^{1} \\
&= \frac{6}{81} + \frac{27}{81} - \left(\frac{36}{81} - \frac{16}{81}\right) = \frac{13}{81}
\end{aligned}
$$



Exercise 10.

1. *Find the volume of the region bounded above by the paraboloid* $z = x^2 + y^2$ *and below by the triangle enclosed by the lines* $y = x, x = 0$, *and* $x + y = 2$ *in the* $xy-$*plane.*
2. *Find the volume of the solid that is bounded above by the cylinder* $z = x^2$ *and below by the region enclosed by the parabola* $y = 2 - x^2$ *and the line* $y = x$ *in the* $xy-$*plane.*
3. *Find the volume of the solid whose base is the region in the* $xy-$*plane that is bounded by the parabola* $y = 4 - x^2$ *and the line* $y = 3x$, *while the top of the solid is bounded by the plane* $z = x + 4$.

Solution for (1.) in Exercise 10

$$
\begin{aligned}
V = \int_0^1 \int_x^{2-x} (x^2+y^2)dy\ dx &= \int_0^1 [x^2y + \frac{y^3}{3})]_x^{2-x} dx \\
&= \int_0^1 [2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}] dx \\
&= [\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}]_0^1 \\
&= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) \\
&= \frac{4}{3}
\end{aligned}
$$

Solution for (2.) in Exercise 10

$$\begin{aligned}
V = \int_{-2}^{1}\int_{x}^{2-x^2} x^2 dy\ dx &= \int_{-2}^{1} [x^2 y]_x^{2-x^2} dx \\
&= \int_{-2}^{1} (2x^2 - x^4 - x^3) dx = [\frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4]_{-2}^{1} \\
&= \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4}\right) - \left(-\frac{16}{3} + \frac{32}{5} - \frac{16}{4}\right) \\
&= \left(\frac{40}{60} - \frac{12}{60} - \frac{15}{60}\right) - \left(-\frac{320}{60} + \frac{384}{60} - \frac{340}{60}\right) \\
&= \frac{63}{20}
\end{aligned}$$

Solution for (3.) in Exercise 10

$$
\begin{aligned}
V = \int_{-4}^{1}\int_{3x}^{4-x^2}(x+4)dy\ dx &= \int_{-4}^{1}[xy+4y]_{3x}^{4-x^2}dx \\
&= \int_{-4}^{1}[x(4-x^2)+4(4-x^2)-3x^2-12x]dx \\
&= \int_{-4}^{1}(-x^3-7x^2-8x+16x)dx \\
&= \left[-\frac{1}{4}x^4-\frac{7}{3}x^3-4x^2+16x\right]_{-4}^{1} \\
&= \left(-\frac{1}{4}-\frac{7}{3}+12\right)-\left(\frac{64}{3}-64\right) \\
&= \frac{157}{3}-\frac{1}{4}=\frac{625}{12}
\end{aligned}
$$

Exercise 11.

1. *Find the volume of the solid in the first octant bounded by the coordinate planes, the plane* $x = 3$*, and the parabolic cylinder* $z = 4 - y^2$.
2. *Find the volume of the solid cut from the first octant by the surface* $z = 4 - x^2 - y$.
3. *Find the volume of the solid cut from the square column* $|x| + |y| \leq 1$ *by the planes* $z = 0$ *and* $3x + z = 3$.
4. *Find the volume of the solid bounded on the front and back by the planes* $x = \pm\pi/3$*, on the sides by the cylinders* $y = \pm \sec x$*, above by the cylinder* $z = 1 + y^2$*, and below by the* $xy-$*plane.*

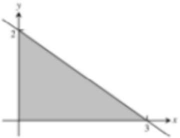
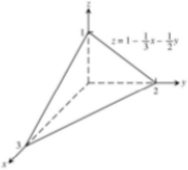
Solution for the Exercise 11

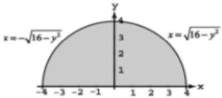
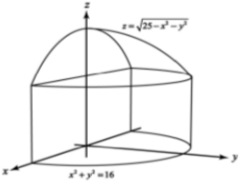
1. $V = \int_0^2 \int_0^3 (4 - y^2) dx \ dy = \int_0^2 [4x - y^2 x]_0^3 dy = \int_0^2 (12 - 3y^2) dy = [12y - y^3]_0^2 = 24 - 8 = 16$
2. $V = \int_0^2 \int_0^{4-x^2} (4 - x^2 - y) dy \ dx = \int_0^2 \left[(4 - x^2)y - \frac{y^2}{2}\right]_0^{4-x^2} dx = \int_0^2 \frac{1}{2}(4 - x^2)^2 dx = \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx = \left[8x - \frac{4}{3}x^3 + \frac{1}{10}x^5\right]_0^2 = 16 - \frac{32}{3} + \frac{32}{10} = \frac{480 - 320 + 96}{30} = \frac{128}{15}$
3. $V = \int_{-1}^0 \int_{-x-1}^{x+1} (3 - 3x) dy \ dx + \int_0^1 \int_{x-1}^{1-x} (3 - 3x) dy \ dx = 6 \int_{-1}^0 (1 - x^2) dx + 6 \int_0^1 (1 - x)^2 dx = 4 + 2 = 6$
4. $V = 4 \int_0^{\pi/3} \int_0^{\sec \ x} (1 + y^2) dy \ dx = 4 \int_0^{\pi/3} \left[y + \frac{y^3}{3}\right]_0^{\sec \ x} dx = 4 \int_0^{\pi/3} \left(\sec \ x + \frac{\sec^3 x}{3}\right) dx = \frac{2}{3}[7 \ln \ |\sec \ x + \tan \ x| + \sec \ x \ \tan \ x]_0^{\pi/3} = \frac{2}{3}\left[7 \ \ln\left(2 + \sqrt{3}\right) + 2\sqrt{3} \ \right]$

Exercise 12.

In the following exercises, sketch the region of integration and the solid whose volume is given by the double integral.

1. $$\int_0^3 \int_0^{2-2x/3} \left(1 - \frac{1}{3}x - \frac{1}{2}y\right) dy\ dx$$
2. $$\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \sqrt{25 - x^2 - y^2} dx\ dy$$

1. 
y
2
3
x

z
1
$z = 1 - \frac{1}{3}x - \frac{1}{2}y$
y
2
3
x

2. 
y
$x = -\sqrt{16 - y^2}$
$x = \sqrt{16 - y^2}$
4
3
2
1
-4 -3 -2 -1 1 2 3 4
x

z
$z = \sqrt{25 - x^2 - y^2}$
y
x
$x^2 + y^2 = 16$

Exercise 13.

Improper double integrals can often be computed similarly to improper integrals of one variable. The first iteration of the following improper integrals is conducted just as if they were proper integrals. One then evaluates an improper integral of a single variable by taking appropriate limits. Evaluate the following improper integrals as iterated integrals.

1. $\displaystyle\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y}\, dy\ dx$
2. $\displaystyle\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)(y^2+1)}\, dx\ dy$
3. $\displaystyle\int_0^\infty \int_0^\infty x e^{(x+2y)}\, dx\ dy$

Solution for the Exercise 13

1. $\displaystyle\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y} dy \ dx = \int_1^\infty [\frac{\ln y}{x^3}]_{e^{-x}}^1 dx = \int_1^\infty -\left(\frac{-x}{x^3}\right) dx = -\lim_{b\to\infty} [\frac{1}{x}]_1^b =$
$\displaystyle -\lim_{b\to\infty}\left(\frac{1}{b}-1\right) = 1$
2. $\displaystyle\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)(y^2+1)} dx \ dy = 2\int_0^\infty \left(\frac{2}{y^2+1}\right)\left(\lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0\right) dy =$
$\displaystyle 2\pi \lim_{b\to\infty} \int_0^b \frac{1}{y^2+1} dy = 2\pi\left(\lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0\right) = (2\pi)(\frac{\pi}{2}) = \pi^2$
3. $\displaystyle\int_0^\infty \int_0^\infty x e^{-(x+2y)} dx \ dy = \int_0^\infty e^{-2y} \lim_{b\to\infty} [-xe^{-x} - e^{-x}]_0^b dy =$
$\displaystyle\int_0^\infty e^{-2y} \lim_{b\to\infty} (-be^{-b} - e^{-b} + 1) dy = \int_0^\infty e^{-2y} dy = \frac{1}{2}\lim_{b\to\infty}(-e^{-2b}+1) = \frac{1}{2}$

Approximating Integrals with Finite Sums

In the following exercise, approximate the double integrals of $f(x, y)$ over the region R partitioned by the given vertical lines $x = a$ and horizontal lines $y = c$. In each subrectangle, use (x_k, y_k) as indicated for your approximation. $\displaystyle\iint_R f(x, y)\, dA \approx \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$.

Exercise 14.

$f(x, y) = x + y$ over the region R bounded above by the semicircle $y = \sqrt{1 - x^2}$ and below by the x-axis, using the partition $x = -1, -1/2, 0, 1/4, 1/2, 1$ and $y = 0, 1/2, 1$ with (x_k, y_k) the lower left corner in the kth subrectangle (provided the subrectangle lies within R)

Solution for the Exercise 14

$$\begin{aligned}\iint_R f(x,y)dA &\approx \frac{1}{4}f(-\frac{1}{2},0)+\frac{1}{8}f(0,0)+\frac{1}{8}f(\frac{1}{4},0)\\ &= \frac{1}{4}(-\frac{1}{2})+\frac{1}{8}(0+\frac{1}{4})\\ &= -\frac{3}{32}\end{aligned}$$

Exercise 15.

1. Circular sector : *Integrate* $f(x, y) = \sqrt{4 - x^2}$ *over the smaller sector cut from the disk* $x^2 + y^2 \leq 4$ *by the rays* $\theta = \pi/6$ *and* $\theta = \pi/2$.
2. Unbounded region : *Integrate* $f(x, y) = 1/\left[\left(x^2 - x\right)(y - 1)^{2/3}\right]$ *over the infinite rectangle* $2 \leq x < \infty, 0 \leq y \leq 2$.
3. Noncircular cylinder : *A solid right (noncircular) cylinder has its base* R *in the* $xy-$ *plane and is bounded above by the paraboloid* $z = x^2 + y^2$. *The cylinder's volume is*

$$V = \int_0^1 \int_0^y \left(x^2 + y^2\right) dx\ dy\ + \int_1^2 \int_0^{2-y} \left(x^2 + y^2\right) dx\ dy.$$

Sketch the base region R *and express the cylinder's volume as a single iterated integral with the order of integration reversed. Then evaluate the integral to find the volume.*

Solution for (1.) in Exercise 15

The ray $\theta = \frac{\pi}{6}$ meets the circle $x^2 + y^2 = 4$ at the point $(\sqrt{3}, 1) \Rightarrow$ the ray is represented by the line $y = \frac{x}{\sqrt{3}}$. Thus,

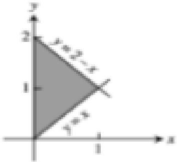
$$\begin{aligned}
\iint_R f(x,y)dA &= \int_0^{\sqrt{3}} \int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2}dy\ dx \\
&= \int_0^{\sqrt{3}} \left[(4-x^2) - \frac{x}{\sqrt{3}}\sqrt{4-x^2}\right] dx \\
&= \left[4x - \frac{x^3}{3} + \frac{(4-x^2)^{3/2}}{3\sqrt{3}}\right]_0^{\sqrt{3}} \\
&= \frac{20\sqrt{3}}{9}
\end{aligned}$$

Solution for (2.) in Exercise 15

$$
\begin{aligned}
\int_2^\infty \int_0^2 \frac{1}{(x^2-x)(y-1)^{2/3}}\,dy\ dx &= \int_2^\infty \Big[\frac{3(y-1)^{1/3}}{(x^2-x)}\Big]_0^2 dx \\
&= \int_2^\infty \left(\frac{3}{x^2-x}+\frac{3}{x^2-x}\right) dx \\
&= 6\int_2^\infty \frac{dx}{x(x-1)} \\
&= 6\lim_{b\to\infty}\int_2^b \left(\frac{1}{x-1}-\frac{1}{x}\right) dx \\
&= 6\lim_{b\to\infty}[\ln(x-1)-\ln\ x]_2^b \\
&= 6\lim_{b\to\infty}[\ln\ (b-1)-\ln\ b-\ln\ 1+\ln\ 2] \\
&= 6\Big[\lim_{b\to\infty}\ln\left(1-\frac{1}{b}\right)+\ln\ 2\Big] = 6\ \ln\ 2
\end{aligned}
$$

Solution for (3.) in Exercise 15

$$
\begin{aligned}
V = \int_0^1 \int_x^{2-x} (x^2 + y^2) dy \ dx &= \int_0^1 [x^2 y + \frac{y^3}{3}]_x^{2-x} dx \\
&= \int_0^1 [2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}] dx \\
&= [\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}]_0^1 \\
&= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - (0 - 0 - \frac{16}{12}) \\
&= \frac{4}{3}
\end{aligned}
$$



Exercise 16.

1. Converting to a double integral : *Evaluate the integral* $\int_0^2 \left(\tan^{-1}\pi x - \tan^{-1} x\right) dx$. *(Hint: Write the integrand as an integral.)*
2. Maximizing a double integral : *What region R in the $xy-$ plane maximizes the value of* $\iint_R \left(4 - x^2 - 2y^2\right) dA$? *Give reasons for your answer.*

1.

$$
\begin{aligned}
\int_0^2 (\tan^{-1}\pi\ x - \tan^{-1} x)dx &= \int_0^2 \int_x^{\pi x} \frac{1}{1+y^2} dy\ dx \\
&= \int_0^2 \int_{y/x}^{y} \frac{1}{1+y^2} dx\ dy + \int_2^{2x} \int_{y/x}^{2} \frac{1}{1+y^2} dx\ dy \\
&= \int_0^2 \frac{1 - \frac{1}{x} y}{1+y^2} dy + \int_2^{2x} \frac{(2 - \frac{y}{x})}{1+y^2} dy \\
&= \left(\frac{\pi - 1}{2\pi}\right) [\ln\ (1+y^2)]_0^2 + [2\tan^{-1} y + \frac{1}{2\pi} \ln(1+y^2)]_2^{2x} \\
&= 2\tan^{-1} 2\pi - 2\tan^{-1} 2 - \frac{1}{2\pi} \ln(1+4\pi^2) + \frac{\ln 5}{2}
\end{aligned}
$$

2. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative. These criteria are met by the points (x, y) such that $4 - x^2 - 2y^2 \geq$ *or* $x^2 + 2y^2 \leq 4$, which is the ellipse $x^2 + 2y^2 = 4$ together with its interior.

Exercise 17.

1. Minimizing a double integral : *What region R in the $xy-$plane minimizes the value of $\iint_R (x^2 + y^2 - 9)\, dA$? Give reasons for your answer.*
2. *Is it possible to evaluate the integral of a continuous function $f(x, y)$ over a rectangular region in the $xy-$ plane and get different answers depending on the orer of integration? Give reasons for your answer.*

Solution for the Exercise 17

1. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where the integrand is positive. These criteria are met by the points (x, y) such that $x^2 + y^2 - 9 \leq 0$ *or* $x^2 + y^2 \leq 9$, which is the closed disk of radius 3 centered at the origin.
2. No, it is not possible. By Fubini's theorem, the two orders of integration must give the same result.

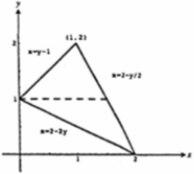
Exercise 18.

How would evaluate the double integral of a continuous funtion $f(x, y)$ *over the region* R *in the* $xy-$ *plane enclosed by the triangle with vertices* $(0, 1), (2, 0)$, *and* $(1, 2)$? *Give reasons for your answer.*

One way would be to partition R into two triangles with the line $y = 1$. The integral of f over R could then be written as a sum of integrals that could be evaluated by integrating first with respect to x and then with respect to y:

$$\iint_R f(x, y)dA = \int_0^1 \int_{2-2y}^{2-(y/2)} f(x, y)dx\ dy + \int_1^2 \int_{y-1}^{2-(y/2)} f(x, y)dx\ dy.$$

Partitioning R with the line $x = 1$ would let us write the integral of f over R as a sum of iterated integrals with order $dy\ dx$.



Exercise 19.

1. Unbounded region : *Prove that* $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dx\ dy =$
$\lim\limits_{b\to\infty}\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}dx\ dy = 4\left(\int_0^{\infty} e^{-x^2}dx\right)^2.$
2. Improper double integral : *Evaluate the improper integral*
$\int_0^3\int_0^3 \frac{x^2}{(y-1)^{2/3}}dy\ dx$

Solution for (1.) in Exercise 19

$$\begin{aligned}
\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\,dx\ dy &= \int_{-b}^{b}\int_{-b}^{b} e^{-y^2}e^{-x^2}\,dx\ dy \\
&= \int_{-b}^{b} e^{-y^2}\Big(\int_{-b}^{b} e^{-e^2}\,dx\Big)dy \\
&= \Big(\int_{-b}^{b} e^{-x^2}\,dx\Big)\Big(\int_{-b}^{b} e^{-y^2}\,dy\Big) \\
&= \Big(\int_{-b}^{b} e^{-x^2}\,dx\Big)^2 \\
&= \Big(2\int_{0}^{b} e^{-x^2}\,dx\Big)^2 \\
&= 4\Big(\int_{0}^{b} e^{-x^2}\,dx\Big)^2.
\end{aligned}$$

Taking limits as $b \to \infty$ gives the stated result.

Solution for (2.) in Exercise 19

$$
\begin{aligned}
\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}}\,dy\ dx &= \int_0^3 \int_0^1 \frac{x^2}{(y-1)^{2/3}}\,dx\ dy \\
&= \int_0^3 \frac{1}{(y-1)^{2/3}} \left[\frac{x^3}{3}\right]_0^1 dy \\
&= \frac{1}{3}\int_0^3 \frac{dy}{(y-1)^{2/3}} \\
&= \frac{1}{3}\lim_{b\to 1}\int_0^b \frac{dy}{(y-1)^{2/3}} + \frac{1}{3}\lim_{b\to 1}\int_b^3 \frac{dy}{(y-1)^{2/3}} \\
&= \lim_{b\to 1}[(y-1)^{1/3}]_0^b + \lim_{b\to 1}[(y-1)^{1/3}]_b^3 \\
&= \left[\lim_{b\to 1}(b-1)^{1/3} - (-1)^{1/3}\right] - \left[\lim_{b\to 1}(b-1)^{1/3} - (2)^{1/3}\right] \\
&= (0+1) - (0-\sqrt[3]{2}) = 1+\sqrt[3]{2}
\end{aligned}
$$

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